Senior Seminar: Project 1
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Lagrange's Theorem (Group Theory)

For any finite group G, the order (number of elements) of every subgroup H of G divides the order of G.

Theorem

The order of a subgroup H of group G divides the order of G. group G, a subgroup of H of G, and a subgroup K of H, *(G:K)=(G:H)(H:K)*.

Proof: For any element $\mathbf{x}$ of G, $\{H\mathbf{x} = \mathbf{h} \cdot \mathbf{x} \mid \mathbf{h} \text{ is in } H\}$ defines a right coset of H. By the cancellation law each $\mathbf{h}$ in H will give a different product when multiplied on the left onto $\mathbf{x}$. Thus $H\mathbf{x}$ will have the same number of elements as H.

Lemma: Two right cosets of a subgroup H of a group G are either identical or disjoint.

Proof: Suppose $H\mathbf{x}$ and $H\mathbf{y}$ have an element in common. Then for some elements $\mathbf{h}_1$ and $\mathbf{h}_2$ of H

$$\mathbf{h}_1 \cdot \mathbf{x} = \mathbf{h}_2 \cdot \mathbf{y}$$

Since H is closed this means there is some element $\mathbf{h}_3$ of H such that $\mathbf{x} = \mathbf{h}_3 \cdot y$. This means that every element of $H\mathbf{x}$ can be written as an element of $H\mathbf{y}$ by the correspondence

$$\mathbf{h} \cdot \mathbf{x} = (\mathbf{h} \cdot \mathbf{h}_3) \cdot \mathbf{y}$$

for every $\mathbf{h}$ in H. We have shown that if $H\mathbf{x}$ and $H\mathbf{y}$ have a single element in common then every element of $H\mathbf{x}$ is in $H\mathbf{y}$. By a symmetrical argument it follows that every element of $H\mathbf{y}$ is in $H\mathbf{x}$ and therefore the "two" cosets must be the same coset.

Since every element $\mathbf{g}$ of G is in some coset the elements of G can be distributed among $\mathbf{H}$ and its right cosets without duplication. If k is the number of right cosets and n is the number of elements in each coset then $|G| = kn$.