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RING

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RING

• $DEFINITION : -$

A non-empty set R , equipped with two binary operations called addition and multiplication denoted by (+) and (.) is said to form a ring if the following properties are satisfied :

Properties under Addition :

1. R is closed with respect to addition
i.e., $a, b \in R$, then $a + b \in R$
2. Addition is associative
i.e., $a + (b + c) = (a + b) + c \; \forall \; a, b, c \in R$
3. Addition is commutative
i.e., $a + b = b + a \; \forall \; a, b \in R$

4. Existence of additive identity

i.e., there exist an additive identity in R denoted by in R such that

$0 + a = a = a + 0 \; \forall \; a \in R$

5. Existence of additive inverse

i.e., to each element a in R, there exists an element $-a$ in R such that

$-a + a = 0 = a + (-a)$

Properties under Multiplication :

6. R is closed with respect to multiplication

i.e., if $a, b \in R$, then $a.b \in R$

7. Multiplication is associative

i.e., $a.(b.c) = (a.b).c \; \forall \; a, b, c \in R$

8. Multiplication is distributive with respect to addition

i.e., $\forall \; a, b, c \in R$, $a.(b + c) = a.b + a.c$ [Left distributive law]

And $(b + c).a = b.a + c.a$[Right distributive law]

• REMARK:

Any algebraic structure $(R, +, .)$ is called a ring if $(R, +)$ is an abelian group and R is closed , associative with respect to multiplication and multiplication is distributive with respect to addition.

⊛ TYPES OF RING

1. COMMUTATIVE RING :

A ring in which $a.b = b.a \; \forall \; a, b \in R$ is called commutative ring.

2. RING WITH UNITY :

If in a ring, there exist an element denoted by 1 such that $1.a = a = a.1$ $\forall \; a \in R$ is called a ring with unity element.

The element $1 \in R$ is called the unit element of the ring.

Thus, if R satisfies the all eight properties of ring and also have multiplicative identity, then we define R as ring with identity.

3. NULL RING OR ZERO RING :

The set R consisting of a single element 0 with two binary operations defined by $0 + 0 = 0$ is a ring and is called null ring or zero ring.

Eg. Prove that the set Z of all integers is a ring with respect to addition and multiplication of integers.

Proof:

. Properties under Addition :

1. Closure property: As sum of two integers is also an integer , Z is closed with respect to addition of integers .

2. Associativity: As addition of integers is also an associative composition

$\therefore$, $a + (b + c) = (a + b) + c \; \forall \; a, b, c \in Z$

3. Existence of additive identity: For $0 \in Z$, $0 + a = a = a + 0 \; \forall$ a $\in Z$.

$\therefore$, 0 is additive identity.

4. Existence of additive inverse: For each $a \in Z$ there exist $-a \in Z$ such that $a + (-a) = 0 = (-a) + a$,

where 0 is identity element .

5. Commutative property :
$a + b = b + a \; \forall \; a, b \in Z$

.Properties under Multiplication:

6. Closure property with respect to multiplication: As product of two integers is also an integer
$a.b \in Z \; \forall \; a, b \in Z$

7. Multiplication is associative :
$a.(b.c) = (a.b).c \; \forall \; a, b, c \in Z$

8. Multiplication is distributive with respect to addition:
$\forall \; a, b, c \in Z$, $a.(b + c) = a.b + a.c$
And $(b + c).a = b.a + c.a$

Hence,Z is a ring with respect to addition and multiplication of integers.

▶ Note:

1. As $1.a = a.1 = a$, $\forall\ a \in Z$,
$\therefore$ 1 is a multiplicative identity of Z.

2. As $a.b = b.a$, $\forall\ a, b \in Z$,
$\therefore$ multiplication of integers is commutative .

Hence, Z is a commutative ring with unity.

⊛ *Remark* :

A ring R is said to be Boolean ring if $x^2 = x\ \forall\ x \in R$.

Eg. Prove that a ring R in which $x^2 = x \ \forall \ x \in R$, must be commutative.
OR
Show that a Boolean ring is commutative.

Proof:

Let $x, y \in R \Rightarrow x + y \in R$

By give condition, $(x+y)^2 = x + y \ \forall \ x, y \in R$

$\Rightarrow (x+y)(x+y) = x + y$

$\Rightarrow x.x + x.y + y.x + y.y = x + y$

$x^2 + x.y + y.x + y^2 = x + y$

$\Rightarrow x + x.y + y.x + y = x + y$ [$\because x^2 = x$, $y^2 = y$]

$\Rightarrow x.y + y.x = 0$

$\Rightarrow x.y = -(y.x)$

$x.y = (-y.x)^2$ ....(1)

Again $\forall\ y \in R$, $(y+y)^2 = y+y$
$\Rightarrow (y+y)(y+y) = y+y$
$\Rightarrow y.y + y.y + y.y + y.y = y+y$
$y^2 + y^2 + y^2 + y^2 = y+y$
$\Rightarrow y+y+y+y = y+y$
$\Rightarrow y+y = 0$
$\Rightarrow y = -y$
$\therefore$ from (1), $x.y = (yx)^2$
$x.y = yx$
Thus $x.y = y.x\ \forall\ x, y \in R$
Hence, R must be commutative.

⊛ RINGS WITH OR WITHOUT ZERO DIVISORS:

A ring $(R, +, .)$ is said to be $without\ zero\ divisors$ if for all a, b belong to R $a.b = 0$ that implies either $a = 0$ or $b = 0$
On the other hand, if in a ring R there exists non zero elements a and b such that $a.b = 0$, then R is said to be a $ring\ with\ zero\ divisors.$
Eg.

1. Sets Z, R, C, and Q are without zero divisors rings.
2. The ring $(0, 1, 2, 3, 4, 5,$ +6, ×6) is a ring with zero divisors.

Eg. Prove that the set $\{0, 1, 2, 3, 4, 5\}$ with addition modulo 6 and multiplication modulo 6 as composition is a ring with zero divisors.

Proof :

Let R =$\{0, 1, 2, 3, 4, 5\}$

Properties under addition :

1. Closure law :

As all the entries in the addition composition table are elements of set R is closed w.r.t. addition modulo 6.

2. Associative law :

The composition $+6$ is associative. If a, b, c are any three elements of R then

a +6 $(b$ +6 $c)$= a +6 $(b + c)$

a +6 (b +6 c)= least non-negative remainder when $a + (b + c)$ is divided by 6

$a+6(b+6c)$=least non-negative remainder when $(a+b)+c$ is divided by 6

a +6 (b +6 c)=$(a+b)$ +6 c

a +6 (b +6 c)=(a +6 b) +6 c

3. Existence of identity :

As 0 +6 a = a=a +6 0 $\forall$ $a \in R$

4. Existence of inverse :

From the table , we see that the inverse of $\{0, 1, 2, 3, 4, 5\}$ are $\{0, 5, 4, 3, 2, 1\}$ respectively. Hence , additive inverse exists.

5. Commutative law :

For all $a, b \in R$, we have a +6 b=b+6 a

Properties under multiplication :

6. Closure law for multiplication :

All the entries in the multiplication composition table are element of set R , therefore R is closed with respect to multiplication modulo 6.

7. Associative law for multiplication :

Let $a, b, c \in R$

$\therefore a$ ×6 (b ×6 c) = a ×6 (bc)

a ×6 $(b$ ×6 $c)$ = least non – negative remainder when $a(bc)$ is divided by 6.

a ×6 $(b$ ×6 $c)4$ = least non negative remainder when $(ab)c$ is divided by 6

a ×6 $(b$ ×6 $c)$ = ab ×6 c

a ×6 $(b$ ×6 $c)$ = $(a$ ×6 $b)$ ×6 c

8. Distribution laws :

If a, b, c be any three elements of R , then

a ×6 $(b$ +6 $c)$ = a ×6 $(b + c)$

a ×6 $(b$ +6 $c)$= least non negative remainder when $a(b + c)$ is divided by 6

a ×6 $(b$ +6 $c)$ = least non – negative remainder when $ab + ac$ is divided by 6

a ×6 $(b$ +6 $c)$ = ab +6 $ac)$

a ×6 $(b$ +6 $c)$ = a ×6 $(b$ +6 $c)$

similarly , $(b$ +6 $c)$ ×6 a = $(b$ ×6 $a)$ +6 $(c$ ×6 $a)$

Hence , R is a ring with respect to given compositions.

As $(R, +6, \times 6)$ is ring ,
Now for $2, 3$ R , $2 \times 3 = 0$
i.e., product of two non zero element is equal to the zero element of the ring .
Hence , R is a ring with zero divisors.