Theorem 2.3. DeMorgan's Laws for sets. Let A and B be sets. Then we have

1. $\overline{A \cup B} = \overline{A} \cap \overline{B}$
2. $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Proof. To prove that $\overline{A \cup B} = \overline{A} \cap \overline{B}$, we start by showing that each set is a subset of the other.The definition of a subset states that A is a subset of B if every element $a \in A$ is also an element of B. Since A and B are sets, if $A \subset B$ and $B \subset A$, then A = B.

Suppose $x \in \overline{A \cup B}$, which means $x \notin A \cup B$. Then $x \notin A$ and $x \notin B$. Hence, $x \in \overline{A}$ and $x \in \overline{B}$. This means $x \in \overline{A} \cap \overline{B}$. Thus, $\overline{A \cup B} \subset \overline{A} \cap \overline{B}$. Now suppose, $x \in \overline{A} \cap \overline{B}$. Then $x \in \overline{A}$ and $x \in \overline{B}$. Hence $x \notin A$ and $x \notin B$, which means that $x \notin A \cup B$. Therefore, $x \in \overline{A \cup B}$. Thus proving that $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

To prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$, we start by showing that each set is a subset of the other. Suppose $x \in \overline{A \cap B}$, which means $x \notin A \cap B$. Then $x \notin A$ and $x \notin B$. Hence, $x \in \overline{A}$ and $x \in \overline{B}$. This means $x \in \overline{A} \cup \overline{B}$. Thus, $\overline{A \cap B} \subset \overline{A} \cup \overline{B}$. Now suppose, $x \in \overline{A} \cup \overline{B}$. Then $x \in \overline{A}$ and $x \in \overline{B}$. Hence $x \notin A$ and $x \notin B$, which means that $x \notin A \cap B$. Therefore, $x \in \overline{A \cap B}$. Thus proving that $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

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Theorem 4.4. Let a,b and c $\in \mathbb{Z}$. If a divides b and b divides c then a divides c.

Proof. Assume a divides b and b divides c. Since a divides b, there exists $n_1 \in \mathbb{Z}$ such that an_1=b. Since b divides c, there exists n_2 such that bn_2=c.Since we know the existential statement is true in the universe you can use it to create an instance of an object with the property it describes. So, we let m=$n_1 n_2$.Then

am= $an_1n_2 = bn_2 =$ c

Since am=c, we have shown that a divides c.

■

Theorem 7.11. Suppose that R is a relation on A. Then R is both symmetric and anti-symmetric, if and only if $R \subset Id_A$.

Proof. Assume R is symmetric and anti-symmetric. This means that all ordered pairs (a,b) $\in$ R, there must be a pair (b,a) $\in$ R, and this can only be true when a=b. This means every ordered pair in R is a value a$\in$ A relates to itself. Since the Id_A is a relation that includes every value in A related to itself, R must be a subset of Id_A.

Assume R $\subset$ Id_A. This means that R can only have elements that are also in Id_A. Therefore every element of R is an ordered pair (a,b) $\in$ A where a=b. Since a=b in every element of R, it satisfied the conditions for anti-symmetry. Also, Since (a,b) = (b,a) , R also satisfies the conditions for symmetry.

■

Theorem 10.9. Let x $\neq$ 1 be any real number. For all natural numbers n we have

$\frac{x^n-1}{x-1} = x^{n-1}+x^{n-2}+...+x^2+x+1$

Proof. First, we define set S as the set n$\in \mathbb{N}$ such that $\sum_{k=1}^{n} x^{k-1} = \frac{x^{n-1}}{x-1}$

I will induct on n

Base Case (n=1): $\sum_{k=1}^{1} x^{k-1} = \frac{x^{1}-1}{x-1} = 1$

Inductive Hypothesis: Assume $\sum_{k=1}^{n} x^{k-1} = \frac{x^{n-1}}{x-1}$ holds for n.

Inductive Step: We want to show $\sum_{k=1}^{n+1} x^{k-1} = \frac{x^{n+1}}{x-1}$

$= \frac{x^{n+1}-x^{n}+x^{n-1}}{x-1}$

$= \frac{x(x^{n})-x^{n}+x^{n}-1}{x-1}$

$= \frac{x(x^{n})}{x-1}$-$\frac{x^{n}}{x-1}+\frac{x^{n-1}}{x-1}$

$=\sum_{k=1}^{n} x^{k-1}+\frac{x(x^{n})}{x-1}$-$\frac{x^{n}}{x-1}$

$=\sum_{k=1}^{n} x^{k-1}+x^{n}$

$=1 + x^{1} + x^{2} + ... +x^{n-2}+x^{n-1}+x^{n}$

$=\sum_{k=1}^{n+1} x^{k-1}$

Therefore, S=$\mathbb{N}$.

■

Theorem 3.2. If $\mathcal{F}$ is a family of sets and A $\in \mathcal{F}$ then A $\subset \cup\mathcal{F}$.

Proof. Let x$\in$ A be arbitrary.We know A$\in\mathcal{F}$.Therefore, we know that there exists A $\in\mathcal{F}$ such that x$\in$ A. Therefore by definition of $\cup\mathcal{F}$, which states that the union of $\mathcal{F}$ is the collections of all sets that are elements of $\mathcal{F}$ where there exists x$\in$A for some A$\in\mathcal{F}$, x $\in\cup\mathcal{F}$. Since x is arbitrary, we have shown that A $\subset \cup\mathcal{F}$.

■

Theorem 6.1. If $0 < a < b$ where a and b are real numbers, then $a^2 < b^2$.

Proof. We assume that $b > a > 0$. Since $a > 0$, if we multiply both sides of $b > a$ by a, we get $ab > a^2$. Similarly, since $b > 0$, if we multiply both sides of $b > a$ by b, we get $b^2 > ab$. We can combine $ab > a^2$ and $b^2 > ab$ to get $b^2 > ab > a^2$. Therefore by the transitive property, $b^2 > a^2$. Thus we have shown that if $b > a > 0$, then $b^2 > a^2$.

■