Riemann Rearrangement Thoerem and Proof

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Riemann Rearrangement Theorem:
Given a conditionally convergent real series

$$\sum_{n=1}^{\infty} a_n$$

and a value $M \in \mathbb{R}$, there exists a rearrangement of the series such that $\sum a_{\sigma(n)} = M$.

Proof. Given $\sum a_n$ is conditionally convergent, $\sum |a_n| = \infty$.

Define subsequences[1] $\left(a_{n_j}\right)_{n_j \in A}$ and $(a_{n_k})_{n_k \in B}$ of a_n by $i \in A \Leftrightarrow a_i < 0$ and $i \in B \Leftrightarrow a_i \geq 0$.

Claim: $\sum_{j=1}^{\infty} a_{n_j} = -\infty$ and $\sum_{k=1}^{\infty} a_{n_k} = \infty$. Suppose both series converge. Then by series addition $\sum |a_n| = \sum_{k=1}^{\infty} a_{n_k} - \sum_{j=1}^{\infty} a_{n_j}$ converges. A contradiction. Suppose one series converges and the other series diverges. Then $\sum_{k=1}^{\infty} a_{n_k} + \sum_{j=1}^{\infty} a_{n_j} = \sum a_n$ diverges. Another contradiction.

Now for the construction of permutation σ of $\mathbb{N}$. Let j_1 be the smallest $\mathbb{N}$ such that

$$\sum_{j=1}^{j_1} a_{n_j} < M.$$

Define $\sigma(j) = n_j \in A$, $\forall j \in [1..j_1]$.[2] Let k_1 be the smallest $\mathbb{N}$ such that

$$\sum_{j=1}^{j_1} a_{n_j} + \sum_{k=1}^{k_1} a_{n_k} > M.$$

Define $\sigma(j_1 + k) = n_k \in B$, $\forall k \in [1..k_1]$.

Step 2: Let j_2 be the smallest $\mathbb{N}$ such that

$$\sum_{j=1}^{j_2} a_{n_j} + \sum_{k=1}^{k_1} a_{n_k} < M.$$

Define $\sigma(j + k_1) = n_j \in A$, $\forall j \in (j_1..j_2]$. Let k_2 be the smallest $\mathbb{N}$ such that

$$\sum_{j=1}^{j_2} a_{n_j} + \sum_{k=1}^{k_2} a_{n_k} > M.$$

[1] Is there a less cumbersome way to define these subsequences?

[2] Here the notation [a..b] refers to all the integers from a through b. Also (a..b) is the set of all integers between a and b.

Define $\sigma(j_2 + k) = n_k \in B$, $\forall k \in (k_1..k_2]$.

Continue defining σ as above and it will be a permutation of $\mathbb{N}$ such that the series rearrangement $\sum a_{\sigma(n)}$ will continue to oscillate around M. First by summing, in order, the negative terms from the sequence (a_n) until the last negative term drops it below M. Then by adding to the sum, in order, from the non-negative terms of sequence (a_n) until the last term pushes is over M.

Let $\varepsilon > 0$. By the divergence test $|a_n| \to 0$. Thus $\exists N \in \mathbb{N}$ such that $\forall n \geq N$ $|a_n| < \varepsilon$. Now $\exists i \in \mathbb{N}$ such that $j_i + k_i > N$. Then since

$$\sum_{j=1}^{j_i} a_{n_j} + \sum_{k=1}^{k_i} a_{n_k} > M \geq \sum_{j=1}^{j_i} a_{n_j} + \sum_{k=1}^{k_i - 1} a_{n_k},$$

we have $\forall p \geq j_i + k_i$, $\left|M - \sum_{n=1}^{p} a_{\sigma(n)}\right| < \varepsilon$. Therefore $\sum a_{\sigma(n)} = M$. $\square$