Using the One Dimensional Wave Equation to Represent Electromagnetic Waves in a Vacuum

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Abstract

The differential wave equation can be used to describe electromagnetic waves in a vacuum. In the one dimensional case, this takes the form $\frac{\partial^2\phi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0$. A general function $f(x,t) = x \pm ct$ will propagate with speed c. To represent the properties of electromagnetic waves, however, the function $\phi(x,t) = \phi_0 sin(kx - \omega t)$ must be used. This gives the Electric and Magnetic field equations to be $E(z,t) = \hat{x}E_0 sin(kz - \omega t)$ and $B(z,t) = \hat{y}B_0 sin(kz - \omega t)$. Using this solution as well as Maxwell's equations the relation $\frac{E_0}{B_0} = c$ can be derived. In addition, the average rate of energy transfer can be found to be $\bar{S} = \frac{E_0^2}{2c\mu_0}\hat{z}$ using the poynting vector of the fields.

1 Introduction

In 1861 James Maxwell published a set of equations to describe electromagnetism, which henceforth became known as Maxwell's equations [1]. They are as follows

1. $\nabla \cdot E = \frac{\rho}{\epsilon_0}$
2. $\nabla \cdot B = 0$
3. $\nabla \times B - \mu_0\epsilon_0\frac{\partial E}{\partial t} = \mu_0 J$
4. $\nabla \times E + \frac{\partial B}{\partial t} = 0$

These equations describe the generation and properties of Electric and Magnetic fields. Applying these equations to the one dimensional wave equation reveals that Maxwell's equations can generate planes waves consisting of Electric and Magnetic Fields.

2 Solving the Wave Equation

2.1 Propagation Speed

The speed of light has been experimentally measured to a high degree of accuracy, and thus any solution for the propagation of an electromagnetic wave in a vacuum must propagate with speed c. If $\phi(x, t)$ is our waveform, then the parameters x and t must sum in the form $\phi(x, t) = f(x \pm ct)$ in order to generate a propagation speed of c. In order to track a point on f, the input value must be held constant. Calling our point $f(k)$ means that $k = x \pm ct$, so the x value must change with speed c in order for the quantity k to remain constant. The direction of the wave's propagation is determined by the sign of ct, with negative resulting in a wave moving in the positive x direction and positive resulting in a wave moving in the negative x direction

2.2 Wave Equation

[2] The function that satisfies both Maxwell's equations and the one dimensional wave equation is $\phi(x, t) = \phi_0 sin(kx - \omega t)$. To prove this, it is only necessary to differentiate $\phi(x, t)$ twice with respect to x and t.

First Derivatives $\frac{\partial \phi}{\partial x} = k\phi_0 \cos(kx - \omega t)$, $\frac{\partial \phi}{\partial t} = -\omega\phi_0 \cos(kx - \omega t)$

Second Derivatives $\frac{\partial^2 \phi}{\partial x^2} = -k^2\phi_0 \sin(kx - \omega t)$, $\frac{\partial^2 \phi}{\partial t^2} = -\omega^2\phi_0 \sin(kx - \omega t)$

Using these in the wave equation results in the following:

$$k^2\phi \sin(kx - \omega t) = \frac{\omega^2}{c^2}\phi_0 \sin(kx - \omega t)$$

Eliminating terms that appear on both sides results in:

$$k^2 = \frac{\omega^2}{c^2}$$

Since the speed of a wave is given by $v = \frac{\omega}{k}$ and we know $v = c$, the equation can be rewritten as $k = \frac{\omega}{c}$. k^2 is therefore $\frac{\omega^2}{c^2}$, so $\phi(x, t) = \phi_0 sin(kx - \omega t)$ solves the wave equation.

2.3 Wave Properties

Having the solution be a sine wave entails certain properties. A sine wave takes completes a cycle in 2π radians, so the angular frequency is related to the frequency of the wave by $f = \frac{\omega}{2\pi}$. Since the frequency is the inverse of the period $T = \frac{2\pi}{\omega}$. The sine wave will also complete a cycle by holding time constant and moving one full wavelength. Since this requires 2π radians, k must equal $\frac{2\pi}{\lambda}$, where lambda is the wavelength.

2.4 Energy Conservation

Since we are considering an electromagnetic wave propagating in a perfect vacuum, energy must be conserved. The energy density of an electric field is given by [3] $\frac{1}{2}\epsilon_0 E^2$. The energy density of a magnetic field is given by [3] $\frac{1}{2}\frac{B^2}{\mu_0}$. Summing these gives $E_{den} = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2}\frac{B^2}{\mu_0}$. Substituting in our equation for both the Electric and Magnetic fields yields

$$E_{den} = \frac{1}{2}\epsilon_0 E_0^2 \sin(kx - \omega t)^2 + \frac{1}{2}\frac{B_0^2 \sin(kx - \omega t)^2}{\mu_0}$$

Factoring out the sine portion gives

$$E_{den} = (\frac{1}{2}\epsilon_0 E_0^2 + \frac{1}{2}\frac{B_0^2}{\mu_0}) \sin(kx - \omega t)^2$$

Integrating over one wavelength (respect to x) gives

$$E_\lambda = (\frac{1}{2}\epsilon_0 E_0^2 + \frac{1}{2}\frac{B_0^2}{\mu_0})\frac{\lambda}{2}$$

This result is not dependent on t, and thus energy is not being gained nor lost over time.

2.5 Mathematical Basis

In order to prove that the wave equation satisfies Maxwell's Equations, a proof of the divergence of the curl and the curl of curl vector identities must be established [4].

2.5.1 Divergence of the Curl

Divergence of the curl states that

$$\nabla \cdot (\nabla \times F) = 0. \tag{1}$$

To prove this, we need only to expand the terms.

$$\nabla \times F = \hat{i}(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}) - \hat{j}(\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z}) + \hat{k}(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y})$$

The dot product yields

$$\nabla \cdot (\nabla \times F) = \frac{\partial F_z}{\partial x \partial y} - \frac{\partial F_y}{\partial x \partial z} - \frac{\partial F_z}{\partial y \partial x} + \frac{\partial F_x}{\partial y \partial z} + \frac{\partial F_y}{\partial z \partial x} - \frac{\partial F_x}{\partial z \partial y}$$

The terms can be rearranged to form

$$\nabla \cdot (\nabla \times F) = (\frac{\partial F_z}{\partial x \partial y} - \frac{\partial F_z}{\partial y \partial x}) + (\frac{\partial F_y}{\partial z \partial x} - \frac{\partial F_y}{\partial x \partial z}) + (\frac{\partial F_x}{\partial y \partial z} - \frac{\partial F_x}{\partial z \partial y})$$

as long as all partials are continuous and differentiable, all terms on the right side cancel leaving

$$\nabla \cdot (\nabla \times F) = 0$$

2.5.2 Curl of the Curl

[5] The Curl of the Curl is

$$\nabla \times (\nabla \times F) = -\nabla^2 F + \nabla(\nabla \cdot F) \tag{2}$$

To prove this, an expansion of terms is sufficient

$$\nabla \times F = \hat{i}(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}) - \hat{j}(\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z}) + \hat{k}(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y})$$

$$\begin{aligned}
&\nabla \times (\nabla \times F) = \\
&\qquad \hat{i}(\frac{\partial^2 F_y}{\partial y \partial x} - \frac{\partial^2 F_x}{\partial y^2} + \frac{\partial^2 F_z}{\partial z \partial x} + \frac{\partial^2 F_x}{\partial z^2}) \\
&\qquad - \hat{j}(\frac{\partial^2 F_y}{\partial x^2} - \frac{\partial^2 F_x}{\partial x \partial y} - \frac{\partial^2 F_z}{\partial z \partial y} - \frac{\partial^2 F_y}{\partial z^2}) \\
&\qquad + \hat{k}(-\frac{\partial^2 F_z}{\partial x^2} + \frac{\partial^2 F_x}{\partial x \partial z} - \frac{\partial^2 F_z}{\partial y^2} - \frac{\partial^2 F_y}{\partial y \partial z})
\end{aligned}$$

Next week need to expand $-\nabla^2 F + \nabla(\nabla \cdot F)$

Starting with the term on the right we get

$$\nabla \cdot F = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

Taking the gradient gives

$$\begin{aligned}
&\nabla(\nabla \cdot F) = \\
&\qquad \hat{i}(\frac{\partial^2 F_x}{\partial x^2} + \frac{\partial^2 F_y}{\partial x \partial y} + \frac{\partial^2 F_z}{\partial x \partial z}) + \\
&\qquad \hat{i}(\frac{\partial^2 F_x}{\partial y \partial x} + \frac{\partial^2 F_y}{\partial y^2} + \frac{\partial^2 F_z}{\partial y \partial z}) + \\
&\qquad \hat{i}(\frac{\partial^2 F_x}{\partial z \partial x} + \frac{\partial^2 F_y}{\partial z \partial y} + \frac{\partial^2 F_z}{\partial z^2})
\end{aligned}$$

[6] The Vector Laplacian $\nabla^2 F$ is applied to each component of a vector, yielding

$$\nabla^2 F =$$

$$\hat{i}(\frac{\partial^2 F_x}{\partial x^2} + \frac{\partial^2 F_x}{\partial y^2} + \frac{\partial^2 F_x}{\partial z^2})+$$
$$\hat{j}(\frac{\partial^2 F_y}{\partial x^2} + \frac{\partial^2 F_y}{\partial y^2} + \frac{\partial^2 F_y}{\partial z^2})+$$
$$\hat{k}(\frac{\partial^2 F_z}{\partial x^2} + \frac{\partial^2 F_z}{\partial y^2} + \frac{\partial^2 F_z}{\partial z^2})$$

Subtracting these two equations gives the final result of

$$\nabla(\nabla \cdot F) - \nabla^2 F =$$

$$\hat{i}(\frac{\partial^2 F_y}{\partial y \partial x} - \frac{\partial^2 F_x}{\partial y^2} + \frac{\partial^2 F_z}{\partial z \partial x} - \frac{\partial^2 F_x}{\partial z^2})$$
$$+ \hat{j}(-\frac{\partial^2 F_y}{\partial x^2} + \frac{\partial^2 F_x}{\partial x \partial y} + \frac{\partial^2 F_z}{\partial z \partial y} - \frac{\partial^2 F_y}{\partial z^2})$$
$$+ \hat{k}(-\frac{\partial^2 F_z}{\partial x^2} + \frac{\partial^2 F_x}{\partial x \partial z} - \frac{\partial^2 F_z}{\partial y^2} - \frac{\partial^2 F_y}{\partial y \partial z})$$

This is equivalent to the expression given for the cross product of the cross product, so

$$\nabla \times (\nabla \times F) = -\nabla^2 F + \nabla(\nabla \cdot F)$$

2.6 Satisfying Maxwell's Equations

With the two previously proven identities, we can now show that the Wave Equation satisfies Maxwell's Equations.

2.6.1 Electric Field

[7] Starting with the Fourth equation we have

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

Using $\nabla\times$ on both sides gives

$$\nabla \times (\nabla \times E) = -\frac{\partial \nabla \times B}{\partial t}$$

From identity (2) we know $\nabla \times (\nabla \times E) = -\nabla^2 E + \nabla(\nabla \cdot E)$ Since there are no sources, $\rho = 0$ so the term $\nabla \cdot E = 0$ from the first of Maxwell's equations, making the term $\nabla(\nabla \cdot E) = 0$. We are then left with

$$\nabla \times (\nabla \times E) = -\nabla^2 E$$

. We can rearrange Maxwell's third equation to be

$$\nabla \times B = \mu_0 J + \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$$

since J is 0 in a vacuum, the equation becomes

$$\nabla \times B = \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$$

Substituting both of these terms into the equation

$$\nabla \times (\nabla \times E) = -\frac{\partial \nabla \times B}{\partial t}$$

and canceling the negative signs gives

$$\nabla^2 E = \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial^2 t}$$

Since this is the one dimensional case $\nabla^2 E = \frac{\partial^2 E}{\partial x^2}$ which gives

$$\frac{\partial^2 E}{\partial x^2} = \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial^2 t}$$

which is the form of the wave equation. This also implies that $\mu_0 \epsilon_0 = \frac{1}{c^2}$, which can be rearranged to give $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$.

2.6.2 Magnetic Field

Starting with the third of Maxwell's equations, we have

$$\nabla \times B - \mu_0 \epsilon_0 \frac{\partial E}{\partial t} = \mu_0 J$$

Since $J = 0$ the equation becomes

$$\nabla \times B - \mu_0 \epsilon_0 \frac{\partial E}{\partial t} = 0$$

Rearranging and applying $\nabla \times$ to the equation yields

$$\nabla \times (\nabla \times B) = \mu_0 \epsilon_0 \frac{\partial \nabla \times E}{\partial t}$$

Identity (2) gives $\nabla \times (\nabla \times B) = -\nabla^2 B + \nabla(\nabla \cdot B)$. Maxwell's second equation says $\nabla \cdot B = 0$. Applying this to the identity gives $\nabla \times (\nabla \times B) = -\nabla^2 B$.

From Maxwell's fourth equation, we get $\nabla \times E = -\frac{\partial B}{\partial t}$. Substituting these two equations into our modified form of Maxwell's third equation gives

$$\nabla^2 B = \mu_0 \epsilon_0 \frac{\partial^2 B}{\partial t^2}$$

Since this is the one dimensional case $\nabla^2 B$ is just $\frac{\partial^2 B}{\partial x^2}$ which gives us

$$\frac{\partial^2 B}{\partial x^2} = \mu_0 \epsilon_0 \frac{\partial^2 B}{\partial t^2}$$

This is the form of the wave equation and, like the electric field version, implies that $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$

2.7 Magnetic and Electric Field Waves

In the Wave Equation section we found $\frac{\partial^2 \phi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial x^2} = 0$ was solved by $\phi(x,t) = \phi_0 sin(kx - \omega t)$. Likewise, the Magnetic and Electric fields were shown to take the form $\frac{\partial^2 B}{\partial x^2} = \mu_0 \epsilon_0 \frac{\partial^2 B}{\partial t^2}$ and $\frac{\partial^2 E}{\partial x^2} = \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial^2 t}$ Which yields solutions $B(z,t) = \hat{y} B_0 sin(kz - \omega t)$ and $E(z,t) = \hat{x} E_0 sin(kz - \omega t)$. We know the directions of the E and B fields must be perpendicular in an electromagnetic wave, so the directions $\hat{x}$ and $\hat{y}$ were assigned to give an orientation.

2.7.1 Relationship between E_0 and B_0

By examining Maxwell's third equation, we can derive a relationship between the values of E_0 and B_0. Maxwell's third equation states

$$\nabla \times B - \mu_0 \epsilon_0 \frac{\partial E}{\partial t} = \mu_0 J$$

Taking the proper derivatives and curl we get

$$\nabla \times B = -k\hat{i} B_0 \cos(kz - \omega t)$$

and

$$\frac{\partial E}{\partial t} = -\omega E_0 \cos(kz - \omega t)\hat{k}$$

Substituting these into Maxwell's third equation and using J = 0 we obtain

$$\omega \mu_0 \epsilon_0 \cos(kz - \omega t)\hat{k} = kB_0 \cos(kz - \omega t)\hat{k}$$

Dividing out like terms yields

$$\omega \mu_0 \epsilon_0 E_0 = kB_0$$

which can be rearranged to obtain

$$\frac{E_0}{B_0} = \frac{k}{\omega \mu_0 \epsilon_0}$$

It has been shown that $\omega = ck$ and that $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$ so the equation can be rewritten as

$$\frac{E_0}{B_0} = c$$

2.7.2 Poynting Vector

The Poynting Vector of an Electromagnetic wave is defined as

$$S = \frac{1}{\mu_0} E \times B$$

Carrying out the cross product yields

$$S = \frac{E_0 B_0 \sin^2(kz - \omega t)}{\mu_0}\hat{z}$$

Taking the average intensity of the wave over a period yields

$$\bar{S} = \frac{E_0 B_0}{2\mu_0}\hat{z}$$

since we know the relationship between E_0 and B_0 to be $\frac{E_0}{B_0} = c$ the expression can also be written as
$\bar{S} = \frac{E_0^2}{2c\mu_0}\hat{z}$ or $\bar{S} = \frac{B_0^2 c}{2\mu_0}\hat{z}$

3 Conclusion

It has been shown that the one dimensional wave equation can be used to satisfy Maxwell's equations for both the Electric and Magnetic fields, and that this solution can be used to derive fundamental properties of an electromagnetic waves. The equations for the Electric and Magnetic fields were computed to be $E(z,t) = \hat{x}E_0 sin(kz - \omega t)$ and $B(z,t) = \hat{y}B_0 sin(kz - \omega t)$ respectively. This was proved using Maxwell's equations as well as the curl of the curl and divergence of the curl vector identities. Using Maxwell's third equation it was determined that E_0 and B_0 were related by the equation $\frac{E_0}{B_0} = c$. Furthermore, the poynting vector of the two fields was used to determine the average rate of energy transfer of the waves, which came out to be $\bar{S} = \frac{E_0^2}{2c\mu_0}\hat{z}$. The one dimensional wave equation can thus be seen as a useful representation of electromagnetic waves propagating in a vacuum, and provides a basis for the properties of the wave.

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