Multivariable Calculus

Extra Credit: Proof of the Cauchy-Schwarz Inequality

Due at the beginning of class on Wednesday, September 9, 2015

Theorem (The Cauchy-Schwarz Inequality)
Suppose $\vec{x}, \vec{y} \in \mathbb{R}^n$. Then

$$|\vec{x} \bullet \vec{y}| \leq \|\vec{x}\|\|\vec{y}\|. \tag{1}$$

Proof
First note that, if either $\vec{x}$ or $\vec{y}$ is the zero vector, then $\vec{x} \bullet \vec{y} = 0$ and $\|\vec{x}\|\|\vec{y}\| = 0$. In this case the theorem is trivially true because $|\vec{x} \bullet \vec{y}| = |0| = 0 = \|\vec{x}\|\|\vec{y}\|$.

Suppose, then, that neither $\vec{x}$ nor $\vec{y}$ is the zero vector. We will establish the truth of an inequality equivalent to (1), namely

$$-\|\vec{x}\|\|\vec{y}\| \leq \vec{x} \bullet \vec{y} \leq \|\vec{x}\|\|\vec{y}\|. \tag{2}$$

To do so we validate the right- and left-hand sides of inequality (2) separately.

Step 1 ($\vec{x} \bullet \vec{y} \leq \|\vec{x}\|\|\vec{y}\|$):
The stipulation that neither $\vec{x}$ nor $\vec{y}$ is the zero vector allows for the following definitions of (unit) vectors $\vec{u}$ and $\vec{w}$:

$$\vec{u} = \frac{\vec{x}}{\|\vec{x}\|},\ \vec{w} = \frac{\vec{y}}{\|\vec{y}\|}. \tag{3}$$

Observe that

$$\begin{aligned}
0 &\leq \|\vec{u} - \vec{w}\|^2 \\
&= (\vec{u} - \vec{w}) \bullet (\vec{u} - \vec{w}) \\
&= \vec{u} \bullet \vec{u} + \vec{w} \bullet \vec{w} - 2(\vec{u} \bullet \vec{w}) \\
&= \|\vec{u}\|^2 + \|\vec{w}\|^2 - 2(\vec{u} \bullet \vec{w}) \\
&= 1 + 1 - 2(\vec{u} \bullet \vec{w}) \\
&= 2 - 2(\vec{u} \bullet \vec{w}),
\end{aligned} \tag{4}$$

which implies that $2(\vec{u} \bullet \vec{w}) \leq 2$, or

$$\vec{u} \bullet \vec{w} \leq 1. \tag{5}$$

Substituting the values for $\vec{u}$ and $\vec{w}$ defined by (3) into inequality (5) yields $\frac{\vec{x}}{\|\vec{x}\|} \bullet \frac{\vec{y}}{\|\vec{y}\|} \leq 1$, which implies $\vec{x} \bullet \vec{y} \leq \|\vec{x}\|\|\vec{y}\|$.

Step 2 ($-\|\vec{x}\|\|\vec{y}\| \leq \vec{x} \bullet \vec{y}$):
Begin by defining the unit vector $\vec{s}$ as

$$\vec{s} = \frac{-\vec{x}}{\|-\vec{x}\|} = \frac{-\vec{x}}{\|\vec{x}\|}.$$

Complete this step for extra credit. To do so, go through a similar argument represented by the expressions in (4), but with the vector $\vec{s} - \vec{w}$. Pay close attention to the minus signs as you proceed.

This is where you should begin Step 2.