NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I EXAMINATION 2012-2013

MAS 722– Topics in Pure Mathematics I

December 2012 TIME ALLOWED: 3 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FIVE (5)** questions and comprises **THREE (3)** printed pages.

2. Answer all questions.

3. Answer each question beginning on a **FRESH** page of the answer book.

4. Candidates may use calculators. However, they should write down systematically the steps in the workings.

5. This is an **open book** examination.

Question 1

Let Z be a Zariski-closed set in A^n, the n-dimensional affine space over a field $\mathbb{F}$.

(i) Show that Z is irreducible if and only if the coordinate ring $\mathbb{F}[Z]$ has no zero divisors. (10 marks)

(ii) Give an example, with justification, of an irreducible Z. (5 marks)

Total: 15 marks

Question 2

Let N be a subvariety (i.e. an irreducible Zariski closed subset) of an affine algebraic variety $M \subset A^n$. Show that $\dim N \leq \dim M$.

Total: 20 marks

Question 3

Let $f := f(x, y)$ be an irreducible complex cubic.

(i) Show that the curve $f = 0$ has at most one singular point, and that the multiplicity of this point is two. (10 marks)
(**Hint**. Use the Weierstrass normal form for the cubics.)

(ii) Show that the curve $f = 0$ is either isomorphic to the nodal cubic $y^2 = x^3 + x^2$, or isomorphic to the cuspidal cubic $y^2 = x^3$. (15 marks)

Total: 25 marks

Question 4

A generating set of an ideal $I \subset \mathbb{F}[x_1, \ldots, x_n]$ is called *universal Gröbner basis* if it is a Gröbner basis of I with respect to any term order. Compute a universal Gröbner basis of the ideal $(x - y^2, xy - x) \subset \mathbb{C}[x, y]$.

Total: 20 marks

Question 5

Let S be the set of polynomials of the form g^ℓ, for each homogeneous $g \in \mathbb{C}[x_0, \ldots, x_n]$ of degree k. Identify S with a closed subset of the Veronese variety in $\mathbb{P}^{\binom{k\ell+n}{n}}$.

Total: 20 marks

END OF PAPER